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THERMAL STRESS INTENSITY FACTORS FOR A CRACK IN A STRIP OF A FUNCTIONALLY GRADIENT MATERIAL

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Abstract—A crack in a strip of Functionally Gradient Material mathematically modeled by a nonhomogeneous solid with the prescribed surface temperature is studied. The crack faces are supposed to be completely insulated. It is assumed that all material properties depend only on the coordinate y (perpendicular to the crack faces) in such a way that the properties are some exponential functions of y. By using the Fourier transform, the thermal and mechanical problems are reduced to two systems of singular integral equations, respectively, which are solved numerically. The results show that by selecting the material constants appropriately, the stress intensity factors can be lowered substantially. The crack close to the cooling side of the strip will be more likely to be stable than that close to the heating surface.

1. INTRODUCTION

It is well known that in aerospace engineering, many structural components are subject to severe thermal loadings which give rise to intense thermal stresses in the components especially around cracks and other kinds of defects. The concentration of stresses around defects often results in catastrophe. In recent years, the concept of so-called Functionally Gradient Materials (FGM) has been introduced and applied to the development of structural components. The advantages of FGM materials are that the material can resist high temperatures effectively and, at the same time, thermal stresses in the material could be relaxed significantly (Noda and Tsuji, 1990; Arai *et al.*, 1991). The interest in FGM research is growing rapidly due to these advantages.

From the viewpoints of applied mechanics and heat conduction, FGM materials are nonhomogeneous solids. Thus the nonhomogeneous continuum theories offer the basis for evaluating the mechanical and thermal properties of FGM. There are only a few papers which have studied the nonhomogeneous problem and most of them studied the case of applied mechanical loading. Erdogan and Delale (Delale and Erdogan, 1983, 1988; Erdogan, 1985) solved some problems for nonhomogeneous elastic materials with cracks subjected to mechanical loadings. Noda and Jin (Noda and Jin, 1992; Jin and Noda, 1992a, b) studied the crack problems in nonhomogeneous thermoelastic solids under steady thermal loadings. They found that the thermal stress intensity factors (SIFs) can be lowered by selecting the material constants appropriately.

In this paper, a crack in a strip of a Functionally Gradient Material mathematically modeled by a nonhomogeneous solid with the prescribed surface temperature is studied. The crack faces are supposed to be completely insulated. It is assumed that all material properties depend only on the coordinate y (perpendicular to the crack face) in such a way that the properties are some exponential functions of y. By using the Fourier transform, the thermal and mechanical problems are reduced to two systems of singular integral equations, respectively, which are solved numerically. Detailed results are presented to illustrate the influence of the nonhomogeneity of the material on the temperature distribution and the stress intensity factors.

2. BASIC EQUATIONS

The basic equations of plane thermal stress problems for nonhomogeneous isotropic elastic bodies are the equilibrium equations (in the absence of body forces):

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=0, \quad \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \tag{1}$$

the strain–displacement relations and the compatibility condition:

$$\varepsilon_x=\frac{\partial u}{\partial x}, \quad \varepsilon_y=\frac{\partial v}{\partial y}, \quad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tag{2}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=2\frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial y}, \tag{3}$$

the constitutive law:

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x-\nu\sigma_y)+\alpha T, \\ \varepsilon_y &= \frac{1}{E}(\sigma_y-\nu\sigma_x)+\alpha T, \\ \varepsilon_{xy} &= \frac{2(1+\nu)}{E}\sigma_{xy}, \end{aligned} \tag{4}$$

and the heat conduction equation (steady-state without heat generation):

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=0. \tag{5}$$

In eqns (4) and (5), E, ν and α are the Young's modulus, the Poisson's ratio and the thermal expansion coefficient, respectively, and k is the heat conductivity.

As shown in Fig. 1, consider an infinite strip containing a through crack with its length being $2c$ and denote by (x, y) the rectangular coordinate system with its origin at the middle point of the crack face and x direction along the crack line. It is assumed that uniform temperatures are maintained over the stress-free boundaries, and the crack faces remain completely insulated. The boundary conditions, therefore, are

$$\begin{aligned} &\sigma_{xy}=\sigma_y=0, && y=0, \quad |x|\leqslant c, \\ &\sigma_{xy}=\sigma_y=0, && y=-a \text{ and } b, \quad |x|<\infty, \\ &\sigma_{\alpha\beta}\to 0, && x^2+y^2\to\infty, \end{aligned} \tag{6}$$

$$\begin{aligned} \sigma_{xy}(x,0^+) &= \sigma_{xy}(x,0^-), \quad |x|>c, \\ \sigma_y(x,0^+) &= \sigma_y(x,0^-), \quad |x|>c, \\ u(x,0^+) &= u(x,0^-), \quad |x|>c, \\ v(x,0^+) &= v(x,0^-), \quad |x|>c, \end{aligned} \tag{7}$$

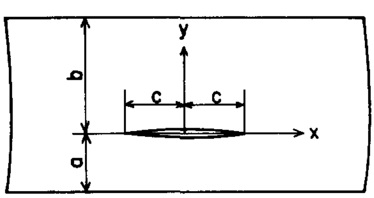

Fig. 1. Crack geometry and coordinates.

for mechanical conditions and

$$
\begin{aligned}
&T = T_a, \quad y = -a, \quad |x| < \infty, \\
&T = 0, \quad y = b, \quad |x| < \infty, \\
&\frac{\partial T}{\partial y} = 0, \quad y = 0, \quad |x| \leqslant c,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&T(x, 0^+) = T(x, 0^-), \quad |x| > c, \\
&\frac{\partial T(x, 0^+)}{\partial y} = \frac{\partial T(x, 0^-)}{\partial y}, \quad |x| > c,
\end{aligned}
\tag{9}
$$

for thermal loading conditions. Here the temperature at $y = b$ is taken to be zero without loss of generality.

FGM materials are usually mixtures of ceramics (which have poorer heat conductivity and lower thermal expansion) and metals (which have higher toughness and better heat conductivity) for resisting high temperatures and relaxing thermal stresses. Hence, from the above thermal loading conditions, it is reasonable to suppose that the material possesses the following nonhomogeneous properties:

$$
\begin{aligned}
&E = E_0\,\mathrm{e}^{\beta y}, \\
&\nu = \nu_0(1+\varepsilon y)\,\mathrm{e}^{\beta y},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\alpha = \alpha_0\,\mathrm{e}^{\gamma y}, \\
&k = k_0\,\mathrm{e}^{\delta y},
\end{aligned}
\tag{11}
$$

where E_0, ν_0, α_0, k_0 and ε, β, γ, δ are material constants. The function $\nu(y)$ given by (10) is subject to the restriction that $0 \leqslant \nu(y) \leqslant 0.5$ for the region of y considered in this paper. The mechanical properties (10) have been used by Delale and Erdogan (1988).

Let $F(x, y)$ be the Airy stress function, then the stresses are expressed in terms of F as

$$
\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}. \tag{12}
$$

By substituting (12) into the compatibility condition (3) through the constitutive law (4) and eqns (10), we obtain:

$$
\nabla^2\nabla^2 F - 2\beta\frac{\partial}{\partial y}(\nabla^2 F) + \beta^2\frac{\partial^2 F}{\partial y^2} + E_0\alpha_0\,\mathrm{e}^{(\beta+\gamma)y}\left(\nabla^2 T + 2\gamma\frac{\partial T}{\partial y} + \gamma^2 T\right) = 0, \tag{13}
$$

where

$$
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
$$

is the two-dimensional Laplacian operator. The substitution of (11) into (5) yields

$$
\nabla^2 T + \delta\frac{\partial T}{\partial y} = 0. \tag{14}
$$

By referring to the dimensionless variables:

$$\sigma^*_{\alpha\beta} = \sigma_{\alpha\beta}/(E_0\alpha_0 T_a),$$
$$F^* = F/(E_0\alpha_0 T_a c^2),$$
$$(u^*, v^*) = (u, v)/(\alpha_0 T_a c),$$
$$\varepsilon^*_{\alpha\beta} = \varepsilon_{\alpha\beta}/(\alpha_0 T_a),$$
$$(x^*, y^*, a^*, b^*) = (x, y, a, b)/c,$$
$$T^* = T/T_a,$$
$$(\beta^*, \gamma^*, \delta^*) = (\beta, \gamma, \delta)c, \tag{15}$$

the governing equations (13), (14) and boundary conditions (6)–(9) have the following dimensionless forms:

$$\nabla^2\nabla^2 F - 2\beta\frac{\partial}{\partial y}(\nabla^2 F) + \beta^2\frac{\partial^2 F}{\partial y^2} + \mathrm{e}^{(\beta+\gamma)y}\left(\nabla^2 T + 2\gamma\frac{\partial T}{\partial y} + \gamma^2 T\right) = 0, \tag{16}$$

$$\nabla^2 T + \delta\frac{\partial T}{\partial y} = 0, \tag{17}$$

$$\begin{aligned} &\sigma_{xy} = \sigma_y = 0, && y = 0, \quad |x| \leqslant 1, \\ &\sigma_{xy} = \sigma_y = 0, && y = -a \text{ and } b, \quad |x| < \infty, \\ &\sigma_{\alpha\beta} \to 0, && x^2 + y^2 \to \infty, \end{aligned} \tag{18}$$

$$\begin{aligned} \sigma_{xy}(x, 0^+) &= \sigma_{xy}(x, 0^-), \quad |x| > 1, \\ \sigma_y(x, 0^+) &= \sigma_y(x, 0^-), \quad |x| > 1, \\ u(x, 0^+) &= u(x, 0^-), \quad |x| > 1, \\ v(x, 0^+) &= v(x, 0^-), \quad |x| > 1, \end{aligned} \tag{19}$$

$$\begin{aligned} T &= 1, \quad y = -a, \quad |x| < \infty, \\ T &= 0, \quad y = b, \quad |x| < \infty, \\ \frac{\partial T}{\partial y} &= 0, \quad y = 0, \quad |x| \leqslant 1, \end{aligned} \tag{20}$$

$$\begin{aligned} T(x, 0^+) &= T(x, 0^-), \quad |x| > 1 \\ \frac{\partial T(x, 0^+)}{\partial y} &= \frac{\partial T(x, 0^-)}{\partial y}, \quad |x| > 1. \end{aligned} \tag{21}$$

Here and in the following, the asterisk * of the dimensionless variables is omitted for simplicity.

3. TEMPERATURE FIELD

The temperature field $T(x, y)$ can be expressed as

$$T(x, y) = T^{(1)}(y) + T^{(2)}(x, y), \tag{22}$$

where $T^{(1)}(y)$ satisfies the following equation and boundary conditions:

$$\frac{d^2T^{(1)}}{dy^2}+\delta\frac{dT^{(1)}}{dy}=0, \tag{23}$$

$$T^{(1)}=1,\quad y=-a,$$
$$T^{(1)}=0,\quad y=b, \tag{24}$$

and $T^{(2)}(x, y)$ is subject to the relations:

$$\nabla^2T^{(2)}+\delta\frac{\partial T^{(2)}}{\partial y}=0, \tag{25}$$

$$T^{(2)}=0,\quad y=-a \text{ and } b,\quad |x|<\infty,$$
$$\frac{\partial T^{(2)}}{\partial y}=-\frac{dT^{(1)}}{dy},\quad y=0,\quad |x|\leqslant 1,$$
$$T^{(2)}(x,0^+)=T^{(2)}(x,0^-),\quad |x|>1$$
$$\frac{\partial T^{(2)}(x,0^+)}{\partial y}=\frac{\partial T^{(2)}(x,0^-)}{\partial y},\quad |x|>1. \tag{26}$$

It is easy to find from (23) and (24) that

$$T^{(1)}(y)=\frac{\exp(-\delta y)-\exp(-\delta b)}{\exp(\delta a)-\exp(-\delta b)}. \tag{27}$$

By applying the Fourier transform to (25), we have:

$$T^{(2)}(x,y)=\int_{-\infty}^{\infty}\{D_1(\xi)\exp(-p_1y)+D_2(\xi)\exp(-p_2y)\}\exp(-ix\xi)\,d\xi,\quad y>0,$$

$$T^{(2)}(x,y)=\int_{-\infty}^{\infty}\{D_3(\xi)\exp(-p_1y)+D_4(\xi)\exp(-p_2y)\}\exp(-ix\xi)\,d\xi,\quad y>0, \tag{28}$$

in the above expressions, $D_i(\xi)$ $(i=1,2,3,4)$ are unknown functions and p_1, p_2 are defined as:

$$p_1=\frac{\delta}{2}-p,\quad p_2=\frac{\delta}{2}+p,\quad p=\sqrt{\xi^2+\frac{\delta^2}{4}}. \tag{29}$$

By substituting (28) into the conditions (26), we can express $D_i(\xi)$ $(i=1,2,3,4)$ as a single function $D(\xi)$ and the resulting temperature expressions are:

$$T^{(2)}(x,y)=\int_{-\infty}^{\infty}\{1-\exp[-2p(b-y)]\}D(\xi)\exp(-p_2y)\exp(-ix\xi)\,d\xi,\quad y>0,$$

$$T^{(2)}(x,y)=\int_{-\infty}^{\infty}\frac{p_2-p_1\exp(-2pb)}{p_1-p_2\exp(-2pa)}\times\{1-\exp[-2p(a+y)]\}D(\xi)\exp(-p_1y)\exp(-ix\xi)\,d\xi,\quad y<0. \tag{30}$$

Introducing the function $\phi(x)$:

$$\phi(x) = \frac{\partial T(x,0^+)}{\partial x} - \frac{\partial T(x,0^-)}{\partial x} = \frac{\partial T^{(2)}(x,0^+)}{\partial x} - \frac{\partial T^{(2)}(x,0^-)}{\partial x}. \tag{31}$$

It is clear from the boundary conditions (26) that

$$\int_{-1}^{1} \phi(t)\,\mathrm{d}t = 0 \quad \text{and} \quad \phi(x) = 0, \qquad x \geqslant 1. \tag{32}$$

By substituting (30) into (31) and using the second condition in (32), we get

$$\phi(x) = \int_{-\infty}^{\infty} -\mathrm{i}\xi D_p(\xi) D(\xi) \exp(-ix\xi)\,\mathrm{d}\xi \tag{33a}$$

and

$$D(\xi) = \frac{i}{2\pi\xi D_p(\xi)} \int_{-1}^{1} \phi(t) \exp(\mathrm{i}\xi t)\,\mathrm{d}t, \tag{33b}$$

where $D_p(\xi)$ is

$$D_p(\xi) = 1 - \exp(-2pb) - \frac{p_2 - p_1 \exp(-2pb)}{p_1 - p_2 \exp(-2pa)} [1 - \exp(-2pa)]. \tag{34}$$

By substituting (22), (27) and (30) into the second condition of (26), with $D(\xi)$ being expressed by (33b), the singular integral equation for $\phi(x)$ is derived as follows:

$$\int_{-1}^{1} \left\{ \frac{1}{t-x} + k(x,t) \right\} \phi(t)\,\mathrm{d}t = \frac{2\pi\delta}{\exp(\delta a) - \exp(-\delta b)}, \quad |x| \leqslant 1, \tag{35}$$

in which the kernel $k(x, t)$ is given by

$$k(x,t) = \int_{0}^{\infty} \left\{ 1 - \frac{2[p_2 - p_1 \exp(-2pb)]}{\xi D_p(\xi)} \right\} \sin[(x-t)\xi]\,\mathrm{d}\xi. \tag{36}$$

For solving the integral equation (35), it is first noted that this kind of equation under the first condition in (32) has the following form of the solution:

$$\phi(x) = \frac{\Phi(x)}{\sqrt{1-x^2}}, \quad |x| \leqslant 1, \tag{37}$$

where $\Phi(x)$ is bounded and continuous on the interval $[-1, 1]$. From the properties of symmetry, it is seen that $\phi(x) = -\phi(-x)$. Then the unknown function may be expressed as follows:

$$\phi(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=1}^{\infty} a_n T_{2n-1}(x), \tag{38}$$

where $T_{2n-1}(x)$ are the Chebyshev polynomials of the first kind. The first condition in (32) is now satisfied automatically. The substitution of (38) into (35) yields the following functional equation for unknown coefficients a_n:

$$\sum_{n=1}^{\infty} a_n[\pi U_{2n-2}(x)+H_n(x)] = \frac{2\pi\delta}{\exp(\delta a)-\exp(-\delta b)}, \quad |x| \leqslant 1, \tag{39}$$

where $U_{2n-2}(x)$ are the Chebyshev polynomials of the second kind and $H_n(x)$ are given by:

$$H_n(x) = \int_{-1}^{1} \frac{k(x,t)}{\sqrt{1-t^2}} T_{2n-1}(t)\,\mathrm{d}t, \quad |x| \leqslant 1. \tag{40}$$

To solve eqn (39), both sides of (39) are expanded into series of Chebyshev polynomials of the first kind. By comparing the coefficients and truncating the series at the Nth term, we obtain

$$\sum_{n=1}^{N} a_n[F_{mn}+G_{mn}] = R_m, \quad m = 1,2,\ldots,N, \tag{41}$$

where

$$F_{mn} = \begin{cases} 1, & 1 \leqslant m \leqslant n, \\ 0, & m > n, \end{cases}$$

$$G_{mn} = \frac{1}{\pi^2}\int_{-1}^{1} \frac{H_n(x)T_{2m-2}(x)\,\mathrm{d}x}{\sqrt{1-x^2}},$$

$$R_1 = \frac{2\delta}{\exp(\delta a)-\exp(-\delta b)}, \quad R_m = 0, \quad 2 \leqslant m \leqslant N. \tag{42}$$

Once $\phi(x)$ is obtained, the temperature field can be easily calculated.

4. THERMAL STRESSES

4.1. *Airy stress function*

Considering the temperature expressions (27) and (30), the governing equation for the Airy function F becomes:

$$\nabla^2\nabla^2F-2\beta\frac{\partial}{\partial y}(\nabla^2F)+\beta^2\frac{\partial^2F}{\partial y^2} =$$
$$-\int_{-\infty}^{\infty}\{[(2\gamma-\delta)p_1-\gamma^2]\exp[-2p(b-y)]-[(2\gamma-\delta)p_2-\gamma^2]\}D(\xi)$$
$$\times\exp[(\beta+\gamma-p_2)y-ix\xi]\,\mathrm{d}\xi+\exp[(\beta+\gamma)y][\gamma^2\exp(-\delta b)$$
$$-(\delta-\gamma)^2\exp(-\delta y)]/[\exp(\delta a)-\exp(-\delta b)], \quad y > 0, \tag{43a}$$

$$\nabla^2\nabla^2F-2\beta\frac{\partial}{\partial y}(\nabla^2F)+\beta^2\frac{\partial^2F}{\partial y^2} =$$
$$-\int_{-\infty}^{\infty}\frac{p_2-p_1\exp(-2pb)}{p_1-p_2\exp(-2pa)}\{[-(2\gamma-\delta)p_1+\gamma^2]$$
$$+[(2\gamma-\delta)p_2-\gamma^2]\exp[-2p(a+y)]\}D(\xi)\exp[(\beta+\gamma-p_1)y-ix\xi]\,\mathrm{d}\xi$$
$$+\exp[(\beta+\gamma)y][\gamma^2\exp(-\delta b)-(\delta-\gamma)^2\exp(-\delta y)]/[(\exp(\delta a)-\exp(-\delta b)], \quad y < 0. \tag{43b}$$

The general solution of (43) can be expressed as

$$F(x,y)=\int_{-\infty}^{\infty}\{(B_1+B_2y)\exp(2sy)+(B_3+B_4y)\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty}\{C_{11}\exp(2py)+C_{12}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi-f_p(y),\quad y>0,\tag{44a}$$

$$F(x,y)=\int_{-\infty}^{\infty}\{(A_1+A_2y)\exp(2sy)+(A_3+B_4y)\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty}\{C_{21}+C_{22}\exp(-2py)\}\exp[(\beta+\gamma-p_1)y-ix\xi]\,d\xi-f_p(y),\quad y<0;\tag{44b}$$

in the above expressions, $A_i(\xi)$ and $B_i(\xi)(i=1,2,3,4)$ are unknown functions, s_1, s_2 are

$$s_1=-\frac{\beta}{2}-s,\quad s_2=-\frac{\beta}{2}+s,\quad s=\sqrt{\xi^2+\frac{\beta^2}{4}}\tag{45}$$

and $C_{ij}(i,j=1,2)$ and $f_p(y)$ are given by

$$C_{11}(\xi)=[(\beta+\gamma-p_1)(\gamma-p_1)-\xi^2]^{-2}[(2\gamma-\delta)p_1-\gamma^2]\exp(-2pb)D(\xi),$$
$$C_{12}(\xi)=[(\beta+\gamma-p_2)(\gamma-p_2)-\xi^2]^{-2}[-(2\gamma-\delta)p_2+\gamma^2]D(\xi),$$
$$C_{12}(\xi)=[(\beta+\gamma-p_1)(\gamma-p_1)-\xi^2]^{-2}[-(2\gamma-\delta)p_1+\gamma^2]\frac{p_2-p_1\exp(-2pb)}{p_1-p_2\exp(-2pa)}D(\xi),$$
$$C_{22}(\xi)=[(\beta+\gamma-p_2)(\gamma-p_2)-\xi^2]^{-2}[(2\gamma-\delta)p_2-\gamma^2]\frac{p_2-p_1\exp(-2pb)}{p_1-p_2\exp(-2pa)}\exp(-2pa)D(\xi),$$
$$f_p(y)=\left\{\frac{\exp[(\beta+\gamma-\delta)y]}{(\beta+\gamma-\delta)^2}-\frac{\exp[(\beta+\gamma)y-\delta b]}{(\beta+\gamma)^2}\right\}\Big/\{\exp(\delta a)-\exp(-\delta b)\}.\tag{46}$$

4.2. *Stresses*

By substituting the Airy function (44) into (12), the following stresses are obtained:

(1) $0<y<b$

$$\sigma_x=\int_{-\infty}^{\infty}\{[-2s_1B_2+s_1^2(B_1+B_2y)]\exp(2sy)$$
$$+[-2s_2B_4+s_2^2(B_3+B_4y)]\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty}\{(\beta+\gamma-p_1)^2C_{11}\exp(2py)+(\beta+\gamma-p_2)^2C_{12}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi-f_p''(y),$$

$$\sigma_y=\int_{-\infty}^{\infty}-\xi^2\{(B_1+B_2y)\exp(2sy)+(B_3+B_4y)\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty}-\xi^2\{C_{11}\exp(2py)+C_{12}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi,$$

$$\sigma_{xy}=\int_{-\infty}^{\infty}i\xi\{[B_2-s_1(B_1+B_2y)]\exp(2sy)+[B_4-s_2(B_3+B_4y)]\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty}i\xi\{(\beta+\gamma-p_1)C_{11}\exp(2py)+(\beta+\gamma-p_2)C_{12}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi;\tag{47}$$

(2) $-a < y < 0$

$$\sigma_x = \int_{-\infty}^{\infty} \{[-2s_1A_2+s_1^2(A_1+A_2y)]\exp(2sy) + [-2s_2A_4+s_2^2(A_3+A_4y)]\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty} \{(\beta+\gamma-p_1)^2C_{21}\exp(2py)+(\beta+\gamma-p_2)^2C_{22}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi - f_p''(y),$$

$$\sigma_y = \int_{-\infty}^{\infty} -\xi^2\{(A_1+A_2y)\exp(2sy)+(A_3+A_4y)\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty} -\xi^2\{C_{21}\exp(2py)+C_{22}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi,$$

$$\sigma_{xy} = \int_{-\infty}^{\infty} i\xi\{[A_2-s_1(A_1+A_2y)]\exp(2sy)+[A_4-s_2(A_3+A_4y)]\}\exp(-s_2y-ix\xi)\,d\xi$$
$$-\int_{-\infty}^{\infty} i\xi\{(\beta+\gamma-p_1)C_{21}\exp(2py)+(\beta+\gamma-p_2)C_{22}\}\exp[(\beta+\gamma-p_2)y-ix\xi]\,d\xi. \quad (48)$$

4.3. *Displacement*

By substituting (2) into (4) and considering (10) and (15), we obtain

$$\frac{\partial u}{\partial x} = [-\nu_0(1+\varepsilon y)\sigma_y+\exp(-\beta y)\sigma_x]+\exp(\gamma y)T,$$
$$\frac{\partial v}{\partial y} = [-\nu_0(1+\varepsilon y)\sigma_x+\exp(-\beta y)\sigma_y]+\exp(\gamma y)T,$$
$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = 2[\nu_0(1+\varepsilon y)+\exp(-\beta y)]\sigma_{xy}. \quad (49)$$

Denoting the jumps of displacements across the line $y = 0$ by $[u]$ and $[v]$, then from (49) and the boundary conditions (18) and (19), we can deduce

$$\frac{\partial[u]}{\partial x} = [\sigma_x]+[T],$$
$$\frac{\partial^2[v]}{\partial x^2} = -\left[\frac{\partial}{\partial y}\{\exp(-\beta y)\sigma_x\}\right]-\gamma[T], \quad (50)$$

where $[\sigma_{\alpha\beta}]$ and $[T]$ denote the jumps in stresses and temperature across $y = 0$, respectively. By substituting (22), (27), (30) and (47), (48) into (50) and integrating the second relation about x, we obtain:

$$\frac{\partial[u]}{\partial x} = \int_{-\infty}^{\infty} \{s_1^2(B_1-A_1)-2s_1(B_2-A_2)+s_2^2(B_3-A_3)-2s_2(B_4-A_4)$$
$$-(\beta+\gamma-p_1)^2(C_{11}-C_{21})-(\beta+\gamma-p_2)^2(C_{12}-C_{22})+D_p(\xi)D(\xi)\}\exp(-ix\xi)\,d\xi,$$
$$\frac{\partial[v]}{\partial x} = \int_{-\infty}^{\infty} -\frac{i}{\xi}\{-(\beta+s_1)s_1^2(B_1-A_1)+(3s_1^2+2\beta s_1)(B_2-A_2)$$
$$-(\beta+s_2)s_2^2(B_3-A_3)+(3s_2^2+2\beta s_2)(B_4-A_4)-(\beta+\gamma-p_1)^2(\gamma-p_1)(C_{11}-C_{21})$$
$$-(\beta+\gamma-p_2)^2(\gamma-p_2)(C_{12}-C_{22})+\gamma D_p(\xi)D(\xi)\}\exp(-ix\xi)\,d\xi. \quad (51)$$

4.4. *Integral equations*

We now introduce two functions $\psi_1(t)$ and $\psi_2(t)$:

$$\psi_1(x) = \frac{\partial[u]}{\partial x}, \quad \psi_2(x) = \frac{\partial[v]}{\partial x}. \tag{52}$$

From the conditions (19), we know that

$$\int_{-1}^{1} \psi_i(t)\,\mathrm{d}t = 0, \quad i = 1,2 \tag{53}$$

and

$$\psi_i(x) = 0, \quad i = 1,2, \quad x \geqslant 1. \tag{54}$$

By using the Fourier inverse transform to (51) and noting the definition (52) and condition (54), we can obtain

$$\begin{aligned} &s_1^2(B_1-A_1)-2s_1(B_2-A_2)+s_2^2(B_3-A_3)-2s_2(B_4-A_4) \\ &\qquad -f_9(\xi) = \frac{1}{2\pi}\int_{-1}^{1} \psi_1(t)\exp(-\mathrm{i}\xi t)\,\mathrm{d}t \\ &-(\beta+s_1)s_1^2(B_1-A_1)+(3s_1^2+2\beta s_1)(B_2-A_2)-(\beta+s_2)s_2^2(B_3-A_3) \\ &+(3s_2^2+2\beta s_2)(B_4-A_4)-f_{10}(\xi) = \frac{\mathrm{i}\xi}{2\pi}\int_{-1}^{1} \psi_2(t)\exp(-\mathrm{i}\xi t)\,\mathrm{d}t, \end{aligned} \tag{55}$$

where $f_9(\xi)$ and $f_{10}(\xi)$ are given in Appendix B. By substituting (47) and (48) into the boundary conditions (18) and (19), six relations satisfied by the eight unknowns $A_i(\xi)$, $B_i(\xi)$ $(i = 1,2,3,4)$ can be obtained, see eqn (A1). We also get the following equations from (18) and (19):

$$\begin{aligned} &\int_{-\infty}^{\infty} \xi^2\{A_1+A_3-f_7(\xi)\}\exp(-ix\xi)\,\mathrm{d}\xi = 0, \quad |x| \leqslant 1, \\ &\int_{-\infty}^{\infty} \mathrm{i}\xi\{A_2-s_1A_1+A_4-s_2A_3-f_8(\xi)\}\exp(-ix\xi)\,\mathrm{d}\xi = 0, \quad |x| \leqslant 1, \end{aligned} \tag{56}$$

where $f_7(\xi)$ and $f_8(\xi)$ are given in Appendix B. Combining (55) and (56), with the help of (A1), the integral equations for $\psi_1(x)$ and $\psi_2(x)$ are obtained as follows:

$$\int_{-1}^{1} \sum_{j=1,2}^{2} \left[\frac{\delta_{ij}}{t-x}+k_{ij}(x,t)\right]\psi_j(t)\,\mathrm{d}t = 2L_i(x), \quad i = 1,2, \quad |x| \leqslant 1, \tag{57}$$

in which

$$L_1(x) = 2\pi\int_{-\infty}^{\infty} \mathrm{i}\xi l_1(\xi)\exp(-ix\xi)\,\mathrm{d}\xi, \quad |x| \leqslant 1,$$

$$L_2(x) = -2\pi\int_{-\infty}^{\infty} \xi^2 l_2(\xi)\exp(-ix\xi)\,\mathrm{d}\xi, \quad |x| \leqslant 1,$$

$$l_1(\xi) = -\frac{H_{11}(\beta g_1+2g_2)+H_{12}(s_2-s_1)(s_2g_1-g_2)}{(s_2-s_1)^3} - G_1,$$

$$l_2(\xi) = -\frac{H_{21}(\beta g_1+2g_2)+H_{22}(s_2-s_1)(s_2g_1-g_2)}{(s_2-s_1)^3}-G_2, \tag{58}$$

$$k_{11}(x,t) = \int_0^\infty (1+4\xi f_{11})\sin(x-t)\xi\,\mathrm{d}\xi,$$

$$k_{22}(x,t) = \int_0^\infty (1+4\xi^2 f_{22})\sin(x-t)\xi\,\mathrm{d}\xi,$$

$$k_{12}(x,t) = \int_0^\infty 4\xi f_{12}\cos(x-t)\xi\,\mathrm{d}\xi,$$

$$k_{21}(x,t) = -\int_0^\infty 4\xi^2 f_{21}\cos(x-t)\xi\,\mathrm{d}\xi, \quad -1 \leqslant x, \quad t \leqslant 1, \tag{59}$$

in (58) and (59), $g_i(\xi)$, $G_i(\xi)$ $(i = 1,2)$ and $H_{ij}(\xi)$, $f_{ij}(\xi)$ $(i,j = 1,2)$ are given in Appendix B.

According to Erdogan *et al.* (1973), the integral equations of the form (57) and (59) have the following form solutions:

$$\psi_i(x) = \frac{\Psi_i(x)}{\sqrt{1-x^2}}, \quad i = 1,2, \quad |x| \leqslant 1, \tag{60}$$

where $\Psi_i(x)$ are continuous bounded functions in the interval $[-1,1]$.

5. STRESS INTENSITY FACTORS

In order to obtain the stress intensity factors, we first derive the expressions of singular stresses ahead of the crack. The singularity of stresses is only due to the asymptotic nature of the integrands in (47) and (48) for large values of the integration variables ξ. Considering this and with the help of (55), (60), (A1) and the asymptotic formula (Gradshteyn and Ryzhik, 1965):

$$\int_{-1}^{1}\frac{\Psi_j(t)}{\sqrt{1-t^2}}\exp(\mathrm{i}\xi t)\,\mathrm{d}t = \sqrt{\frac{\pi}{2|\xi|}}\left\{\Psi_j(1)\exp\left[\mathrm{i}\left(\xi-\frac{\pi|\xi|}{4\xi}\right)\right]\right.$$
$$\left.-\Psi_j(-1)\exp\left[-\mathrm{i}\left(\xi-\frac{\pi|\xi|}{4\xi}\right)\right]+O\left(\frac{1}{|\xi|}\right)\right\}, \tag{61}$$

we can obtain the expressions for the singular stresses from (47) near the crack tip $(1,0)$ just ahead of the tip as follows:

$$\{\sigma_{xy},\sigma_y\} = -\frac{\{\Psi_1(1),\Psi_2(1)\}}{4\sqrt{2\pi}}\int_0^\infty\frac{1}{\sqrt{\xi}}\sin\left[\xi(x-1)+\frac{\pi}{4}\right]\mathrm{d}\xi, \quad x > 1. \tag{62}$$

Making use of the well-known formulae

$$\int_0^\infty\frac{1}{\sqrt{\xi}}\{\sin(c\xi),\cos(c\xi)\}\,\mathrm{d}\xi = \left\{\sqrt{\frac{\pi}{2c}},\sqrt{\frac{\pi}{2c}}\right\}, \quad c > 0, \tag{63}$$

the singular stresses ahead of the crack tip can be obtained:

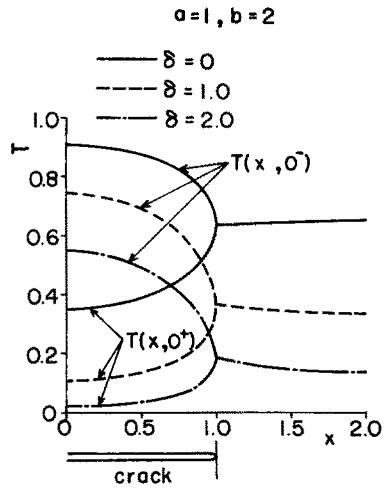

Fig. 2. The temperatures on the crack faces and the crack extended line ($x > 1, y = 0$), $(a, b) = (1, 2)$.

$$\{\sigma_y, \sigma_{xy}\} = \frac{1}{\sqrt{2\pi(x-1)}}\{K_{\rm I}, K_{\rm II}\}, \quad y = 0, \quad x > 1, \tag{64}$$

where the dimensionless stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ are

$$K_{\rm I} = -\frac{\sqrt{\pi}}{4}\Psi_2(1),$$

$$K_{\rm II} = -\frac{\sqrt{\pi}}{4}\Psi_1(1). \tag{65}$$

6. NUMERICAL RESULTS AND DISCUSSIONS

6.1. *Temperature field*

The temperatures on the crack faces and the crack extended line ($y = 0, x > 1$) for $(a, b) = (1, 2)$ and $(2, 1)$ and different δ are depicted in Figs 2 and 3. The following two facts can be observed. Firstly, the temperatures decrease with the increase of the nonhomogeneous parameter δ, and secondly, the temperature jumps across the crack faces are smaller when the crack is closer to the cooling side of the strip, i.e. $(a, b) = (2, 1)$, than the heating side, i.e. $(a, b) = (1, 2)$.

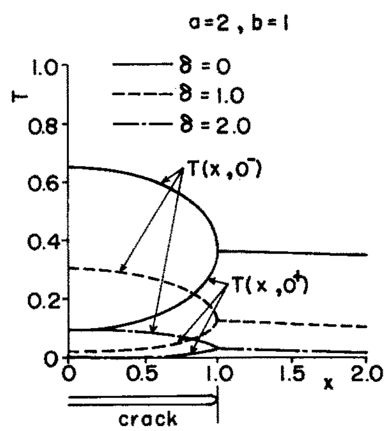

Fig. 3. The temperatures on the crack faces and the crack extended line ($x > 1, y = 0$), $(a, b) = (2, 1)$.

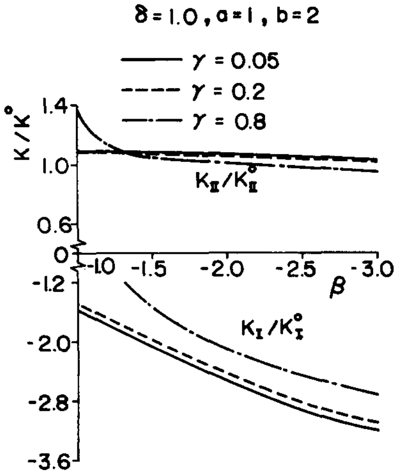

Fig. 4. The effect of the nonhomogeneous parameters β, γ and δ on SIFs, $(a, b) = (1, 2)$.

6.2. *The effect of the nonhomogeneity of the material on the stress intensity factors*

The stress intensity factors (SIFs) can be obtained once we get the solutions of the integral equations (57). The numerical technique used here is the same as that in Section 3 for the temperature field.

In the homogeneous case (β, γ and δ are all zero), the calculated values of K_I and K_{II} are $K_I^0 = 0.01174$, $K_{II}^0 = -0.05037$ for $(a, b) = (1, 2)$ and $K_I^0 = -0.01174$, $K_{II}^0 = -0.05037$ for $(a, b) = (2, 1)$, respectively. It should be noted that no results of this homogeneous problem have been reported in the literature.

The effect of the material nonhomogeneous parameters, β, δ and γ on the SIFs for $(a, b) = (1, 2)$ are depicted in Figs 4–5. It is clear that the Mode II SIF K_{II} is relatively insensitive to β, γ or δ, whereas the Mode I SIF K_I is dramatically affected by β. K_I vary from the positive values K_I^0 at $\beta = 0$ to negative values about three times those of $-K_I^0$ at $\beta = -3$. This means that the comprehensive stress intensity near the crack tip will probably decrease with the decrease of β. In Figs 6 and 7, the effect of β, δ and γ on the maximum of the cleavage stress σ_θ near the crack tip which roughly characterized the nature of crack initiation under mixed mode fracture are shown. It appears that the maximum values can be lowered significantly by decreasing β. The direction of crack initiation, θ_m, at which the cleavage stress σ_θ around the crack tip takes the maximum value is depicted in Fig. 8. Since the direction is not sensitive to the parameter γ, only the results of $\gamma = 0.2$ are presented. It appears that the crack will tend to grow toward the less stiff side of the solid and in a direction nearly perpendicular to the crack face. For $(a, b) = (2, 1)$, i.e. the crack is closer

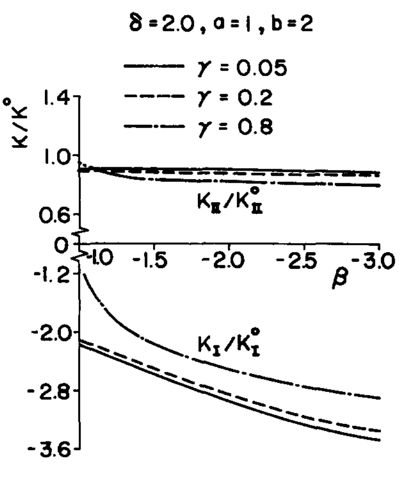

Fig. 5. The effect of the nonhomogeneous parameters β, γ and δ on SIFs, $(a, b) = (1, 2)$.

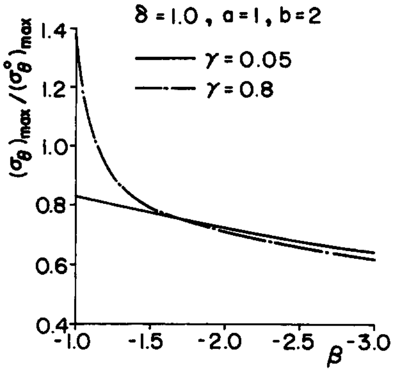

Fig. 6. The effect of β, γ and δ on the maximum cleavage stress σ_θ, $(a, b) = (1, 2)$.

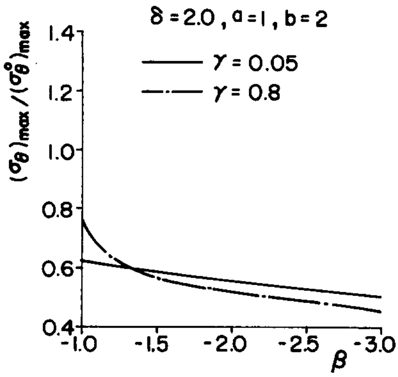

Fig. 7. The effect of β, γ and δ on the maximum cleavage stress σ_θ, $(a, b) = (1, 2)$.

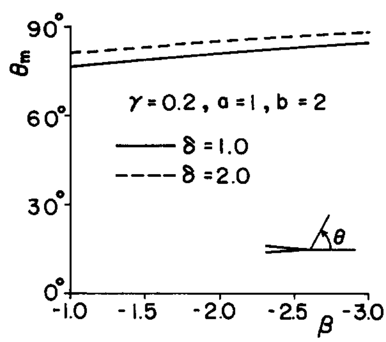

Fig. 8. The effect of β, γ and δ on the direction of crack initiation θ_m, $(a, b) = (1, 2)$. For the homogeneous medium, θ_m is 66.15.

to the cooling side of the strip, the results are depicted in Figs 9–13. In this case, the maximum of the cleavage stress σ_θ can be lowered more substantially than the case of $(a, b) = (1, 2)$ by decreasing β. It seems from these results that the crack closer to the cooling side of the strip will be more stable than that closer to the heating surface.

From Figs 4 and 5 and Figs 9 and 10, we know that K_I is negative for the non-homogeneous material in the range of β considered in this paper. In fact, there exist values of β at which K_I becomes zero and when β becomes smaller (negative larger), K_I becomes negative so that the contact of the crack faces would occur. The results presented here without considering this effect may not be exact but would be more conservative. Since the contact of the crack faces will increase the friction between the faces and make heat transfer across the crack faces easier, the stress intensity factors would be lowered by these two factors.

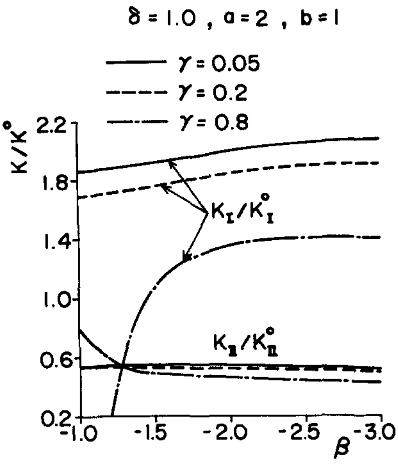

Fig. 9. The effect of the nonhomogeneous parameters β, γ and δ on SIFs, $(a, b) = (2, 1)$.

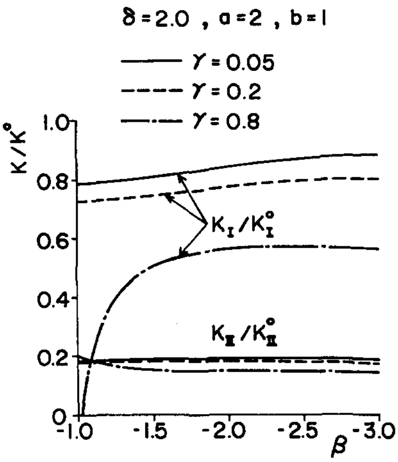

Fig. 10. The effect of the nonhomogeneous parameters β, γ and δ on SIFs, $(a, b) = (2, 1)$.

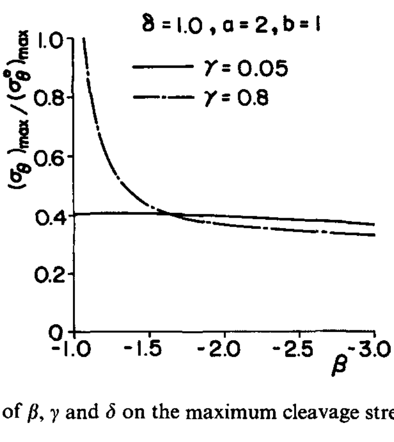

Fig. 11. The effect of β, γ and δ on the maximum cleavage stress σ_θ, $(a, b) = (2, 1)$.

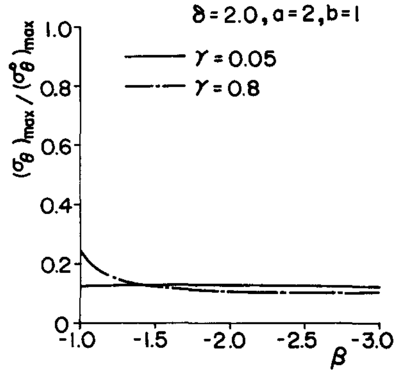

Fig. 12. The effect of β, γ and δ on the maximum cleavage stress σ_θ, $(a, b) = (2, 1)$.

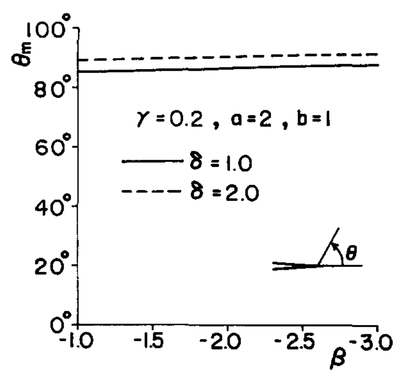

Fig. 13. The effect of β, γ and δ on the direction of crack initiation θ_m, $(a, b) = (2, 1)$. For the homogeneous medium, θ_m is 75.03.

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REFERENCES

Arai, Y., Kobayashi, H. and Tamura, M. (1991). Elastic–plastic thermal stress analysis for optimum design of FGM. *Proceedings of FGM'91*, pp. 19–30. Kawasaki, Japan.

Delale, F. and Erdogan, F. (1983). The crack problem for a nonhomogeneous plane. *J. Appl. Mech.* **50,** 609–614.

Delale, F. and Erdogan, F. (1988). On the mechanical modeling of the interfacial region in bonded half-planes. *J. Appl. Mech.* **55,** 317–324.

Erdogan, F. (1985). The crack problem for bonded nonhomogeneous materials under antiplane shear loading. *J. Appl. Mech.* **52,** 823–828.

Erdogan, F., Gupta, G. D. and Cook, T. S. (1973). Numerical solution of singular integral equations. In *Mechanics of Fracture, Vol. 1* (Edited by G. C. Sih), pp. 368–425. Noordhoff, Leyden.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Tables of Integrals, Series and Products.* Academic Press, New York.

Jin, Z.-H. and Noda, N. (1992a). An internal crack parallel to the boundary of a nonhomogeneous half plane under thermal loading. *Int. J. Engng Sci.* (to appear).

Jin, Z.-H. and Noda, N. (1992b). Minimization of thermal stress intensity factors for a crack in a metal-ceramics mixture. *The 2nd International Symposium on Functionally Gradient Materials.* 2–4 November, San Francisco.

Noda, N. and Jin, Z.-H. (1992). Steady thermal stresses in an infinite nonhomogeneous elastic solid containing a crack. *J. Thermal Stresses* (to appear).

Noda, N. and Tsuji, T. (1990). Steady thermal stresses in a plate of Functionally Gradient Material. *Proceedings of the 1st Int. Symposium on Functionally Gradient Materials*, pp. 339–344, 8–9 October, Sendai, Japan.

APPENDIX A

The equations satisfied by the unknown functions $A_i(\xi)$ and $B_i(\xi)$ $(i = 1,2,3,4)$:

$$\begin{aligned}
&(B_1+bB_2)\exp(-s_1b)+(B_3+bB_4)\exp(-s_2b) = f_1(\xi),\\
&[B_2-s_1(B_1+bB_2)]\exp(-s_1b)+[B_4-s_2(B_3+bB_4)]\exp(-s_2b) = f_2(\xi),\\
&(A_1-aA_2)\exp(s_1a)+(A_3-aA_4)\exp(s_2a) = f_3(\xi),\\
&[A_2-s_1(A_1-aA_2)]\exp(s_1a)+[A_4-s_2(A_3-aA_4)]\exp(s_2a) = f_4(\xi),\\
&B_1-A_1+B_3-A_3 = f_5(\xi),\\
&B_2-A_2+B_4-A_4-s_1(B_1-A_1)-s_2(B_3-A_3) = f_6(\xi),
\end{aligned} \tag{A1}$$

where the functions $f_i(\xi)$ $(i = 1,2,\ldots,6)$ are given in Appendix B.

APPENDIX B

Functions used in Section 4:

$$\begin{aligned}
h_{11}(\xi) &= -s_1+\exp(-2sa)[s_1+2as_2s],\\
h_{12}(\xi) &= 1-\exp(-2sa)[1-2as(1-as_2)],\\
h_{21}(\xi) &= 1-\exp(-2sa)[1+2as],\\
h_{22}(\xi) &= 2a^2s\exp(-2sa),
\end{aligned} \tag{B1}$$

$$\begin{aligned}
H_{11}(\xi) &= (h_{11}d_{11}+h_{12}d_{21})/D_A,\\
H_{12}(\xi) &= (h_{11}d_{12}+h_{12}d_{22})/D_A,\\
H_{21}(\xi) &= (h_{21}d_{11}+h_{22}d_{21})/D_A,\\
H_{22}(\xi) &= (h_{21}d_{12}+h_{22}d_{22})/D_A,
\end{aligned} \tag{B2}$$

$$\begin{aligned}
d_{11}(\xi) &= \exp[-2s(a+b)][1-2s(a+b)(1-2as)-(1-2as+4a^2s^2)\exp(-2sb)]\\
&\quad +\exp(-2sb)(1+2bs+4b^2s^2)-1,\\
d_{12}(\xi) &= 2s\exp[-2s(a+b)][(a+b)(a-b-2abs)-a^2\exp(-2sb)]+\exp(-2sb)2b^2s,\\
d_{21}(\xi) &= 4s^2\exp[-2s(a+b)][a+b-a\exp(-2sb)]-\exp(-2sb)4b^2s,\\
d_{22}(\xi) &= \exp[-2s(a+b)][(1+2as)(1+2bs-\exp(-2sb)+4b^2s^2]+(1-2bs)\exp(-2sb)-1,\\
D_A(\xi) &= 1-2[1+2(a+b)^2s^2]\exp[-2(a+b)s]+\exp[-4(a+b)s],
\end{aligned} \tag{B3}$$

$$\begin{aligned}
f_{11}(\xi) &= [-\beta H_{11}+s_2(s_1-s_2)H_{12}](s_1-s_2)^{-3},\\
f_{12}(\xi) &= [-2\xi H_{11}-\xi(s_1-s_2)H_{12}](s_1-s_2)^{-3},\\
f_{21}(\xi) &= [-\beta H_{21}+s_2(s_1-s_2)H_{22}](s_1-s_2)^{-3},\\
f_{22}(\xi) &= [-2\xi H_{21}-\xi(s_1-s_2)H_{22}](s_1-s_2)^{-3},
\end{aligned} \tag{B4}$$

$$\begin{aligned}
g_1(\xi) &= s_2^2f_5+2s_2f_6+f_9,\\
g_2(\xi) &= (2s_2+\beta)s_2^2f_5+(3s_2+2\beta)s_2f_6-f_{10},\\
g_3(\xi) &= \exp(-s_2a)[(1-as_2)f_4-as_2^2f_3]-f_8,\\
g_4(\xi) &= \exp(-s_2a)[(1+as_2)f_3+af_4]-f_7,\\
g_5(\xi) &= -\exp(s_1b)[(1-bs_2)f_1-bf_2]+\exp(-s_2a-2sb)[(1+as_2)f_3+af_4]+\exp(-2sb)f_5,\\
g_6(\xi) &= -\exp(s_1b)[s_2f_1+f_2]+\exp(-s_2a-2sb)[s_2f_3+f_4]+\exp(-2sb)[s_2f_5+f_6],\\
g_7(\xi) &= \{(1-2as)\exp[-2s(a+b)]-(1+2bs)\}g_5-2s\{b^2-a^2\exp[(-2s(a+b)]\}g_6,\\
g_8(\xi) &= \{(1+2as)\exp[-2s(a+b)]-(1-2bs)\}g_6-2s\{-1+\exp[(-2s(a+b)]\}g_5,
\end{aligned} \tag{B5}$$

$$\begin{aligned}
G_1(\xi) &= g_3+h_{11}g_7+h_{12}g_8,\\
G_2(\xi) &= g_4+h_{21}g_7+h_{22}g_8,
\end{aligned} \tag{B6}$$

$$\begin{aligned}
f_1(\xi) &= C_{11}\exp[(\beta+\gamma-p_1)b]+C_{12}\exp[(\beta+\gamma-p_2)b],\\
f_2(\xi) &= (\beta+\gamma-p_1)C_{11}\exp[(\beta+\gamma-p_1)b]+(\beta+\gamma-p_2)C_{12}\exp[(\beta+\gamma-p_2)b],\\
f_3(\xi) &= C_{21}\exp[-(\beta+\gamma-p_1)a]+C_{22}\exp[-(\beta+\gamma-p_2)a],\\
f_4(\xi) &= (\beta+\gamma-p_1)C_{21}\exp[-(\beta+\gamma-p_1)a]+(\beta+\gamma-p_2)C_{22}\exp[-(\beta+\gamma-p_2)a],\\
f_5(\xi) &= C_{11}-C_{21}+C_{12}-C_{22},
\end{aligned}$$

$$f_6(\xi) = (\beta+\gamma-p_1)(C_{11}-C_{21})+(\beta+\gamma-p_2)(C_{12}-C_{22}),$$
$$f_7(\xi) = C_{21}+C_{22},$$
$$f_8(\xi) = (\beta+\gamma-p_1)C_{21}+(\beta+\gamma-p_2)C_{22},$$
$$f_9(\xi) = (\beta+\gamma-p_1)^2(C_{11}-C_{21})+(\beta+\gamma-p_2)^2(C_{12}-C_{22})-D_p(\xi)D(\xi),$$
$$f_{10}(\xi) = (\beta+\gamma-p_1)^2(\gamma-p_1)(C_{11}-C_{21})+(\beta+\gamma-p_2)^2(\gamma-p_2)(C_{12}-C_{22})-\gamma D_p(\xi)D(\xi). \quad \text{(B7)}$$

In these functions, C_{11}, C_{21}, C_{22} are given by (46), $D(\xi)$ is given by (33b) and $D_p(\xi)$ is given by (34).